

REAL HYPERSURFACES IN THE COMPLEX PROJECTIVE PLANE ATTAINING EQUALITY IN A BASIC INEQUALITY

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ABSTRACT. We determine non-Hopf hypersurfaces with constant mean curvature in the complex projective plane which attain equality in a basic inequality between the maximum Ricci curvature and the squared mean curvature.

1. STATEMENT OF THE MAIN THEOREM

Let M be a real hypersurface in the complex projective space $\mathbb{C}P^n(4)$ of complex dimension n and constant holomorphic sectional curvature 4. We denote by J the almost complex structure of $\mathbb{C}P^n(4)$. The characteristic vector field on M is defined by $\xi = -JN$ for a unit normal vector field N . If ξ is a principal curvature vector at $p \in M$, then M is said to be *Hopf* at p . If M is Hopf at every point, then M is called a *Hopf hypersurface*. Let \mathcal{H} be the holomorphic distribution defined by $\mathcal{H} = \bigcup_{p \in M} \{X \in T_p M \mid \langle X, \xi \rangle = 0\}$. If \mathcal{H} is integrable and each leaf of its maximal integral manifolds is locally congruent to $\mathbb{C}P^{n-1}(4)$, then M is called a *ruled real hypersurface*, which is a typical example of a non-Hopf hypersurface.

For a Riemannian manifold M , let \overline{Ric} denote the maximum Ricci curvature function on M defined by

$$\overline{Ric}(p) = \max\{S(X, X) \mid X \in T_p M, \|X\| = 1\},$$

where S is the Ricci tensor. On the other hand, the δ -invariant $\delta(2)$ of M is defined by $\delta(2)(p) = (1/2)\tau(p) - \min\{K(\pi) \mid \pi \text{ is a plane in } T_p M\}$, where τ is the scalar curvature of M and $K(\pi)$ is the sectional curvature of π . (For general δ -invariants, see [4] for details.) In the case of $\dim M = 3$, we have $\overline{Ric}(p) = \delta(2)(p)$. Thus, according to Corollary 7 and Theorem 8 in [2], for real hypersurfaces in $\mathbb{C}P^2(4)$ we have the following:

Theorem 1.1 ([2]). *Let M be a real hypersurface in $\mathbb{C}P^2(4)$. Then we have*

$$(1.1) \quad \overline{Ric} \leq \frac{9}{4}\|H\|^2 + 5.$$

If M is a Hopf hypersurface, then the equality in (1.1) holds identically if and only if one of the following two cases occurs:

- (1) *M is an open portion of a geodesic sphere with radius $\pi/4$;*
- (2) *M is an open portion of a tubular hypersurface over a complex quadric curve Q_1 with radius $r = \tan^{-1}((1 + \sqrt{5} - \sqrt{2 + 2\sqrt{5}})/2) = 0.33311971 \dots$.*

The next step is to classify non-Hopf hypersurfaces in $\mathbb{C}P^2(4)$ which satisfy the equality in (1.1) identically. The main theorem of this paper is the following.

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Theorem 1.2. *Let M be a real hypersurface in $\mathbb{CP}^2(4)$ which is non-Hopf at every point. Assume that M has constant mean curvature. Then M satisfies the equality case of (1.1) identically if and only if it is a minimal ruled real hypersurface which is given by $\varpi \circ z$, where $\varpi : S^5 \rightarrow \mathbb{CP}^2(4)$ is the Hopf fibration and*

$$z(u, v, \theta, \psi) = e^{\sqrt{-1}\psi} (\cos u \cos v, \cos u \sin v, (\sin u)e^{\sqrt{-1}\theta})$$

for $-\pi/2 < u < \pi/2$, $0 \leq v, \theta, \psi < 2\pi$.

Remark 1.1. Let M be a real hypersurface in the complex hyperbolic plane $\mathbb{CH}^2(-4)$ of constant holomorphic sectional curvature -4 . Then we have

$$\overline{Ric} \leq \frac{9}{4} \|H\|^2 - 2.$$

The equality sign of the inequality holds identically if and only if M is an open part of the horosphere (see [2] or [3]).

2. PRELIMINARIES

Let M be a real hypersurface in $\mathbb{CP}^n(4)$. Let us denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and $\mathbb{CP}^n(4)$, respectively. The Gauss and Weingarten formulas are respectively given by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + \langle AX, Y \rangle N, \\ \tilde{\nabla}_X N &= -AX\end{aligned}$$

for tangent vector fields X, Y and a unit normal vector field N , where A is the shape operator. The mean curvature vector field H is defined by $H = (\text{Tr} A / (2n - 1))N$. The function $\text{Tr} A / (2n - 1)$ is called the *mean curvature*. If it vanishes identically, then M is called a *minimal hypersurface*.

For any vector field X tangent to M , we denote the tangential component of JX by PX . Then by the Gauss and Weingarten formulas, we have

$$(2.1) \quad \nabla_X \xi = PAX.$$

We denote by R the Riemannian curvature tensor of M . Then, the equations of Gauss and Codazzi are respectively given by

$$(2.2) \quad \begin{aligned}R(X, Y)Z &= \langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle PY, Z \rangle PX - \langle PX, Z \rangle PY \\ &\quad - 2 \langle PX, Y \rangle PZ + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,\end{aligned}$$

$$(2.3) \quad (\nabla_X A)Y - (\nabla_Y A)X = \langle X, \xi \rangle PY - \langle Y, \xi \rangle PX - 2 \langle PX, Y \rangle \xi.$$

We need the following two lemmas for later use.

Lemma 2.1 ([2]). *Let M be a real hypersurface in $\mathbb{CP}^2(4)$. Then the equality sign in (1.1) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, e_3\}$ at p such that*

- (1) $\langle Pe_1, e_2 \rangle = 0$,
- (2) the shape operator of M in $\mathbb{CP}^2(4)$ at p satisfies

$$(2.4) \quad A = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \alpha + \gamma = \mu.$$

Lemma 2.2 ([5]). *Let M a real hypersurface M in $\mathbb{C}P^n(4)$ with $n \geq 2$. We define differentiable functions α, β on M by $\alpha = \langle A\xi, \xi \rangle$ and $\beta = \|A\xi - \alpha\xi\|$. Then M is ruled if and only if the following two conditions hold:*

- (1) *the set $M_1 = \{p \in M \mid \beta(p) \neq 0\}$ is an open dense subset of M ;*
- (2) *there is a unit vector field U on M_1 , which is orthogonal to ξ and satisfies*

$$(2.5) \quad A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi, \quad AX = 0$$

for an arbitrary tangent vector X orthogonal to both ξ and U .

3. PROOF OF THE MAIN THEOREM

Let M be a real hypersurface in $\mathbb{C}P^2(4)$ which is non-Hopf at every point. Assume that M has constant mean curvature and satisfies the equality case of (1.1) identically.

Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame field described in Lemma 2.1. It follows from (1) of Lemma 2.1 that ξ lies in $\text{Span}\{e_1, e_2\}$. Thus, we may assume that $e_1 = \xi$ and $Pe_2 = e_3$. Then, (2.4) can be rewritten as

$$(3.1) \quad A\xi = (\mu - \gamma)\xi + \beta e_2, \quad Ae_2 = \gamma e_2 + \beta\xi, \quad Ae_3 = \mu e_3.$$

Since ξ is not a principal vector everywhere, we have $\beta \neq 0$ on M . The constancy of the mean curvature implies that μ is constant. By (2.1) and (3.1), we have

$$(3.2) \quad \nabla_{e_2}\xi = \gamma e_3, \quad \nabla_{e_3}\xi = -\mu e_2, \quad \nabla_\xi\xi = \beta e_3.$$

Since $\langle \nabla_{e_i}, e_j \rangle = -\langle \nabla_{e_j}, e_i \rangle$ holds, it follows from (3.2) that

$$\begin{aligned} \nabla_{e_2}e_2 &= \kappa_1 e_3, \quad \nabla_{e_3}e_2 = \kappa_2 e_3 + \mu\xi, \quad \nabla_\xi e_2 = \kappa_3 e_3, \\ \nabla_{e_2}e_3 &= -\kappa_1 e_2 - \gamma\xi, \quad \nabla_{e_3}e_3 = -\kappa_2 e_2, \quad \nabla_\xi e_3 = -\kappa_3 e_2 - \beta\xi \end{aligned}$$

for some smooth functions κ_1, κ_2 and κ_3 .

With respect to the Gauss-Codazzi equations, we are going to state only equations that are useful in this proof.

From the equation (2.3) of Codazzi, we obtain:

- for $X = e_2$ and $Y = \xi$, by comparing the coefficient of e_3 ,

$$(3.3) \quad \beta\kappa_1 + (\mu - \gamma)\kappa_3 = \beta^2 + \gamma^2 - 1.$$

- for $X = e_3$ and $Y = \xi$, by noting that μ is constant and $\beta \neq 0$,

$$(3.4) \quad e_3\beta = \mu^2 - 2\mu\gamma - \kappa_3(\mu - \gamma) + \beta^2 + 1,$$

$$(3.5) \quad e_3\gamma = 2\beta\mu + \beta\gamma - \beta\kappa_3,$$

$$(3.6) \quad \kappa_2 = 0.$$

- for $X = e_2$ and $Y = e_3$, by comparing the coefficient of e_2 ,

$$(3.7) \quad e_3\gamma = -\mu\kappa_1 + \kappa_1\gamma + \beta\gamma + 2\beta\mu.$$

Eliminating $e_3\gamma$ from (3.5) and (3.7) yields

$$(3.8) \quad (\mu - \gamma)\kappa_1 - \beta\kappa_3 = 0.$$

By solving (3.3) and (3.8) for κ_1 and κ_3 , we get

$$(3.9) \quad \kappa_1 = \frac{\beta(\beta^2 + \gamma^2 - 1)}{(\mu - \gamma)^2 + \beta^2},$$

$$(3.10) \quad \kappa_3 = \frac{(\mu - \gamma)(\beta^2 + \gamma^2 - 1)}{(\mu - \gamma)^2 + \beta^2}.$$

By applying the equation (2.2) of Gauss for $X = e_2$, $Y = Z = e_3$, comparing the coefficient of e_2 and using (3.6), we obtain

$$(3.11) \quad e_3\kappa_1 - 2\mu\gamma - \kappa_1^2 - (\gamma + \mu)\kappa_3 - 4 = 0.$$

Substituting (3.9) and (3.10) into (3.4), (3.7) and (3.11), we obtain

$$(3.12) \quad e_3\beta = (\mu - 2\gamma)\mu + \beta^2 + 1 - \frac{(\mu - \gamma)^2(\beta^2 + \gamma^2 - 1)}{(\mu - \gamma)^2 + \beta^2},$$

$$(3.13) \quad e_3\gamma = \beta(\gamma + 2\mu) + \frac{(\gamma - \mu)\beta(\beta^2 + \gamma^2 - 1)}{(\mu - \gamma)^2 + \beta^2},$$

$$(3.14) \quad \begin{aligned} & \{(3\beta^2 + \gamma^2 - 1)((\mu - \gamma)^2 + \beta^2) - 2\beta^2(\beta^2 + \gamma^2 - 1)\}e_3\beta \\ & + \{2\beta\gamma((\mu - \gamma)^2 + \beta^2) + 2(\mu - \gamma)\beta(\beta^2 + \gamma^2 - 1)\}e_3\gamma \\ & - 2\mu\gamma\{(\mu - \gamma)^2 + \beta^2\}^2 - \beta^2(\beta^2 + \gamma^2 - 1)^2 \\ & + (\gamma^2 - \mu^2)(\beta^2 + \gamma^2 - 1)\{(\mu - \gamma)^2 + \beta^2\} - 4\{(\mu - \gamma)^2 + \beta^2\}^2 = 0. \end{aligned}$$

Substituting (3.12) and (3.13) into (3.14) gives

$$(3.15) \quad (\mu - \gamma)f(\beta, \gamma) = 0,$$

where $f(\beta, \gamma)$ is given by the following polynomial.

$$\begin{aligned} f(\beta, \gamma) := & 2\mu\gamma^4 - (4\mu^2 - 1)\gamma^3 + (3\mu^2 + 4\beta^2 - 6)\mu\gamma^2 - \{\mu^4 + (4\beta^2 - 7)\mu^2 - \beta^2 - 1\}\gamma \\ & + (\beta^2 - 2)\mu^3 + (2\beta^4 - 2\beta^2 - 1)\mu. \end{aligned}$$

By (3.15), our discussion is divided into two cases.

Case (a): $\mu - \gamma = 0$. In this case, from (3.8) we obtain $\kappa_3 = 0$. Therefore, by (3.5) and the constancy of μ , we see that $\mu = \gamma = 0$.

Case (b): $f(\beta, \gamma) = 0$. In this case, differentiating $f(\beta, \gamma) = 0$ along e_3 , by using (3.12) and (3.13) we obtain

$$(3.16) \quad \begin{aligned} & 8\mu\gamma^6 - (24\mu^2 - 4)\gamma^5 + (30\mu^2 + 24\beta^2 - 15)\mu\gamma^4 \\ & - \{20\mu^4 + (48\beta^2 + 3)\mu^2 - 8\beta^2 - 3\}\gamma^3 \\ & + \{7\mu^5 + (36\beta^2 + 45)\mu^3 + (24\beta^4 - 10\beta^2 - 2)\mu\}\gamma^2 \\ & - \{\mu^6 + (12\beta^2 + 44)\mu^4 + (24\beta^4 + 19\beta^2 + 2)\mu^2 - 4\beta^4 - 3\beta^2 + 1\}\gamma \\ & + (\beta^2 + 13)\mu^5 + (6\beta^4 + 19\beta^2 + 1)\mu^3 + (8\beta^6 + 5\beta^4 - 2\beta^2 + 1)\mu = 0. \end{aligned}$$

The resultant of $f(\beta, \gamma)$ and the LHS of (3.16) with respect to γ is given by

$$(3.17) \quad 202500(\mu^2 - 1)^4\beta^4\mu^6\{4\mu^2\beta^2 + (\mu^2 - 1)^2\}^2.$$

Since $\beta \neq 0$, we have $4\mu^2\beta^2 + (\mu^2 - 1)^2 \neq 0$. By changing the sign of N if necessary, we may assume that $\mu \geq 0$. Thus, from (3.17) we get $\mu \in \{0, 1\}$. If $\mu = 1$, then equations $f(\beta, \gamma) = 0$ and (3.16) can be reduced to

$$2\beta^2 + 2\gamma^2 + \gamma - 3 = 0,$$

$$8\beta^4 + (16\gamma^2 - 4\gamma + 3)\beta^2 + (\gamma - 1)^2(8\gamma^2 + 12\gamma + 15) = 0,$$

respectively. This system of equations has a unique solution $(\beta, \gamma) = (0, 1)$, which contradicts $\beta \neq 0$. Hence, we have $\mu = 0$. Then, equation $f(\beta, \gamma) = 0$ becomes $\gamma(\beta^2 + \gamma^2 + 1) = 0$, which shows that $\gamma = \alpha = 0$.

From the above argument, the shape operator satisfies

$$A\xi = \beta e_2, \quad Ae_2 = \beta\xi, \quad Ae_3 = 0$$

at each point, where $\beta \neq 0$. By Lemma 2.2 we conclude that M is a minimal ruled real hypersurface. According to [1], it is congruent to the real hypersurface described in Theorem 1.2.

The converse is clear from Lemma 2.1 and (2.5). The proof is finished. ■

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